

ON THE EQUATION $f^n(z) + g^n(z) = e^{\alpha z + \beta}$

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ABSTRACT. We describe meromorphic solutions to the equations $f^n(z) + (f')^n(z) = e^{\alpha z + \beta}$ and $f^n(z) + f^n(z+c) = e^{\alpha z + \beta}$ ($c \neq 0$) over the complex plane \mathbf{C} for integers $n \geq 1$.

This paper is devoted to the description of meromorphic solutions to the following functional equation

$$f^n(z) + g^n(z) = e^{\alpha z + \beta}, \quad (1)$$

where $g(z) = f'(z)$ or $g(z) = f(z+c)$ for $\alpha, \beta, c (\neq 0) \in \mathbf{C}$, when $n \geq 1$.

In particular, when $\alpha = \beta = 0$, then (1) is reduced to the following well-known Fermat-type functional equation, initialed by Gross [8, 9, 10] and Baker [1],

$$f^n(z) + g^n(z) = 1. \quad (2)$$

Below, we summarize all possible solutions to (2) (see theorem 2.3 in Han [11]).

Proposition 1. *For nonconstant meromorphic solutions f and g to the functional equation (2), one has that (A) when $n = 2$, the only solutions are $f = \frac{2\omega}{1+\omega^2}$ and $g = \frac{1-\omega^2}{1+\omega^2}$ for a nonconstant meromorphic function ω ; (B) when $n = 3$, the only solutions are $f = \frac{1}{2\mathfrak{p}(h)} \left(1 + \frac{\sqrt{3}}{3}\mathfrak{p}'(h)\right)$ and $g = \frac{\eta}{2\mathfrak{p}(h)} \left(1 - \frac{\sqrt{3}}{3}\mathfrak{p}'(h)\right)$ for a nonconstant entire function h and a cube root η of unity, where \mathfrak{p} denotes the Weierstrass \mathfrak{p} -function; (C) when $n \geq 4$, there are no such solutions.*

Via $\omega = \tan\left(\frac{h}{2}\right)$, we have $f = \frac{2\omega}{1+\omega^2} = \sin(h)$ and $g = \frac{1-\omega^2}{1+\omega^2} = \cos(h)$ in case (A) for $n = 2$ are the only entire solutions to the functional equation (2) for an entire function h . Moreover, $\mathfrak{p}(z)$, the Weierstrass elliptic \mathfrak{p} -function with periods ω_1 and ω_2 , is defined to be

$$\mathfrak{p}(z; \omega_1, \omega_2) := \frac{1}{z^2} + \sum_{\mu, \nu \in \mathbf{Z}; \mu^2 + \nu^2 \neq 0} \left\{ \frac{1}{(z + \mu\omega_1 + \nu\omega_2)^2} - \frac{1}{(\mu\omega_1 + \nu\omega_2)^2} \right\},$$

which is an even function and satisfies, after appropriately choosing ω_1 and ω_2 ,

$$(\mathfrak{p}')^2 = 4\mathfrak{p}^3 - 1. \quad (3)$$

For meromorphic solutions of partial differential equations similar to (1), we refer the reader to Li [13, 14], Chang and Li [4], Han [11], and the references therein.

Below, we assume the familiarity with the basics of Nevanlinna theory [16] of meromorphic functions in \mathbf{C} such as the first and second main theorems, and the standard notations such as the characteristic function $T(r, f)$, the proximity function $m(r, f)$, and the counting functions $N(r, f)$ (counting multiplicity) and $\bar{N}(r, f)$ (ignoring multiplicity). $S(r, f)$ denotes a quantity satisfying $S(r, f) = o(T(r, f))$ as $r \rightarrow \infty$, except possibly on a set of finite logarithmic measure which is not necessarily the same at each occurrence.

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First, consider meromorphic solutions to $f^n + (f')^n = \gamma^n$ when $n \geq 4$ and $\gamma \neq 0$. According to the conclusions of proposition 1, both $\frac{f}{\gamma}$ and $\frac{f'}{\gamma}$ are constant. Assume $f = c_1\gamma$ and $f' = c_2\gamma$ to see $c_1\gamma' = c_2\gamma$ with $c_1^n + c_2^n = 1$. If $c_1 = 0$, $f \equiv 0$ and thus $\gamma \equiv 0$. So, $c_1 \neq 0$, and if $c_2 = 0$, f is a constant and so is γ . When $c_1c_2 \neq 0$, then γ cannot have zeros and poles, and one sees $\gamma^n(z) = e^{\alpha z + \beta}$ with $\alpha = n\frac{c_2}{c_1}$. This is another reason why we focus on $e^{\alpha z + \beta}$.

Next, for $f^3 + (f')^3 = e^{\alpha z + \beta}$, f must be entire and thus both $\frac{f}{\gamma}$ and $\frac{f'}{\gamma}$ are constant, so that the same conclusion holds as above. Now, for $f^2 + (f')^2 = e^{\alpha z + \beta}$, f must again be entire and $f(z) = e^{\frac{\alpha z + \beta}{2}} \sin(h(z))$ and $f'(z) = e^{\frac{\alpha z + \beta}{2}} \cos(h(z))$ by proposition 1, so that $\frac{\alpha}{2} \tan(h) \equiv 1 - h'$. As $T(r, h') = O(T(r, h)) + S(r, h)$ and $\lim_{r \rightarrow \infty} \frac{T(r, \tan(h))}{T(r, h)} = +\infty$ (see Clunie [6, theorem 2 (i)] that extended Pólya [17]), we see that either $\alpha = 0$ and $h' = 1$, or h is a constant.

Summarizing the preceding discussions leads to the following result.

Theorem 2. *Solutions f to the following differential equation*

$$f^n(z) + (f')^n(z) = e^{\alpha z + \beta} \quad (4)$$

must be entire and are such that (A) when $n = 1$, the general solutions are $f(z) = \frac{e^{\alpha z + \beta}}{\alpha + 1} + ae^{-z}$ for $\alpha \neq -1$ and $f(z) = ze^{-z + \beta} + ae^{-z}$; (B) when $n = 2$, either $\alpha = 0$ and the general solutions are $f(z) = e^{\frac{\beta}{2}} \sin(z + b)$, or $f(z) = de^{\frac{\alpha z + \beta}{2}}$; (C) finally when $n \geq 3$, then the general solutions are $f(z) = de^{\frac{\alpha z + \beta}{n}}$. Here, $\alpha, \beta, a, b, d \in \mathbf{C}$ with $d^n(1 + (\frac{\alpha}{n})^n) = 1$ for $n \geq 1$.

Note when $n \geq 2$, equation (4) may have no solution for $\alpha = ne^{\frac{(2k+1)\pi i}{n}}$, $k = 0, 1, \dots, n-1$. In addition, we refer the reader to Li [15] for some related interesting results.

Now, consider meromorphic solutions $f(z)$ to the following difference equation, with $c \neq 0$,

$$f^n(z) + f^n(z+c) = e^{\alpha z + \beta}. \quad (5)$$

When $n \geq 1$, take $f(z) = c_1 e^{\frac{\alpha z + \beta}{n}}$ and $f(z+c) = c_2 e^{\frac{\alpha z + \beta}{n}}$ to see $c_1 e^{\frac{\alpha c}{n}} = c_2$ with $c_1^n + c_2^n = 1$, inspired by case (C) of proposition 1. Note that $c_1c_2 \neq 0$ and $f(z+c) = e^{\frac{\alpha c}{n}} f(z)$. As a result, all those *trivial* solutions are $f(z) = de^{\frac{\alpha z + \beta}{n}}$ with $d^n(1 + e^{\alpha c}) = 1$ for $n \geq 1$.

Next, we discuss the existence of nontrivial solutions to (5) when $n = 3^1$.

Theorem 3. *There is no solution of finite order to the following difference equation*

$$f^3(z) + f^3(z+c) = e^{\alpha z + \beta}. \quad (6)$$

Here, the order of f is defined to be $\rho(f) := \limsup_{r \rightarrow +\infty} \frac{\log T(r, f)}{\log r}$.

Proof. Via proposition 1, one has

$$f(z) = \frac{1}{2} \frac{\left\{1 + \frac{\sqrt{3}}{3} \mathbf{p}'(h(z))\right\}}{\mathbf{p}(h(z))} e^{\frac{\alpha z + \beta}{3}} \quad \text{and} \quad f(z+c) = \frac{\eta}{2} \frac{\left\{1 - \frac{\sqrt{3}}{3} \mathbf{p}'(h(z))\right\}}{\mathbf{p}(h(z))} e^{\frac{\alpha z + \beta}{3}}. \quad (7)$$

Thus, a routine computation leads to

$$\frac{\eta \left\{1 - \frac{\sqrt{3}}{3} \mathbf{p}'(h(z))\right\}}{\mathbf{p}(h(z))} = \frac{\left\{1 + \frac{\sqrt{3}}{3} \mathbf{p}'(h(z+c))\right\}}{\mathbf{p}(h(z+c))} e^{\frac{\alpha c}{3}}. \quad (8)$$

Assume $\rho(f) < \infty$. Then, from (3) and the first equality in (7), one has

$$\frac{3f^2(z)\mathbf{p}^2(h(z))}{e^{\frac{2}{3}(\alpha z + \beta)}} - \frac{3f(z)\mathbf{p}(h(z))}{e^{\frac{1}{3}(\alpha z + \beta)}} + 1 = \mathbf{p}^3(h(z)). \quad (9)$$

¹Please be reminded that a similar yet simpler approach has been applied for the discussion of meromorphic solutions to $f^3(z) + f^3(z+c) = 1$ in Han and Lü [12].

Notice the estimate (2.7) of Bank and Langley [2] says that

$$T(r, \mathbf{p}) = \frac{\pi}{A} r^2 (1 + o(1)) \quad \text{and} \quad \rho(\mathbf{p}) = 2. \quad (10)$$

Here, A is the area of the parallelogram \mathfrak{S} with vertices $0, \omega_1, \omega_2, \omega_1 + \omega_2$.

Recall $T(r, e^{\alpha z}) = \frac{|\alpha|}{\pi} r (1 + o(1))$. We combine (9) and (10) to observe

$$T(r, \mathbf{p}(h)) \leq 2T(r, f) + \frac{2}{3}T(r, e^{\alpha z}) + O(1), \quad (11)$$

and hence $\rho(\mathbf{p}(h)) < \infty$ as well. By corollary 1.2 of Edrei and Fuchs [7] (see also theorem 1 of Bergweiler [3] for a different and elegant proof), h must be a polynomial.

A side note here is $T(r, \mathbf{p}(h)) = O(r^{2l})$ for some positive integer $l \geq 1$.

Notice when $\mathbf{p}(z_0) = 0$, then $(\mathbf{p}')^2(z_0) = -1$ by (3). Now, write all the zeros of \mathbf{p} by $\{z_j\}_{j=1}^{\infty}$ that satisfy $|z_j| \rightarrow \infty$ as $j \rightarrow \infty$, and assume that $h(a_{j,k}) = z_j$ for $k = 1, 2, \dots, \deg(h)$. Then, we have $(\mathbf{p}')^2(h(a_{j,k})) = (\mathbf{p}')^2(z_j) = -1$.

Suppose there is a subsequence of $\{a_{j,k}\}_{j=1}^{\infty}$ with respect to j such that $\mathbf{p}(h(a_{j,k} + c)) = 0$. Denote this subsequence still by $\{a_{j,k}\}_{j=1}^{\infty}$ and without loss of generality fix the index k below. So, $(\mathbf{p}')^2(h(a_{j,k} + c)) = -1$. Differentiate (8) and use substitution to derive

$$\begin{aligned} & \eta \left\{ 1 - \frac{\sqrt{3}}{3} \mathbf{p}'(h(a_{j,k})) \right\} \mathbf{p}'(h(a_{j,k} + c)) h'(a_{j,k} + c) \\ &= \left\{ 1 + \frac{\sqrt{3}}{3} \mathbf{p}'(h(a_{j,k} + c)) \right\} \mathbf{p}'(h(a_{j,k})) h'(a_{j,k}) e^{\frac{\alpha c}{3}}, \end{aligned}$$

from which we observe one and only one of the following situations appears

$$\begin{cases} \eta \left\{ 1 - i \frac{\sqrt{3}}{3} \right\} h'(a_{j,k} + c) = \left\{ 1 + i \frac{\sqrt{3}}{3} \right\} h'(a_{j,k}) e^{\frac{\alpha c}{3}}, \\ \eta h'(a_{j,k} + c) = -h'(a_{j,k}) e^{\frac{\alpha c}{3}}, \\ \eta \left\{ 1 + i \frac{\sqrt{3}}{3} \right\} h'(a_{j,k} + c) = \left\{ 1 - i \frac{\sqrt{3}}{3} \right\} h'(a_{j,k}) e^{\frac{\alpha c}{3}}. \end{cases}$$

As $h(z)$ and $h(z + c)$ are polynomials of the same leading coefficient, and there are infinitely many $a_{j,k}$'s with $|a_{j,k}| \rightarrow \infty$ when $j \rightarrow \infty$, we would have to conclude

$$\begin{cases} \eta \left\{ 1 - i \frac{\sqrt{3}}{3} \right\} h'(z + c) = \left\{ 1 + i \frac{\sqrt{3}}{3} \right\} h'(z) e^{\frac{\alpha c}{3}}, \\ \eta h'(z + c) = -h'(z) e^{\frac{\alpha c}{3}}, \\ \eta \left\{ 1 + i \frac{\sqrt{3}}{3} \right\} h'(z + c) = \left\{ 1 - i \frac{\sqrt{3}}{3} \right\} h'(z) e^{\frac{\alpha c}{3}}. \end{cases} \quad (12)$$

This is possible only if α, c satisfy $e^{\frac{\alpha c}{3}} = -1, \frac{1}{2} \pm i \frac{\sqrt{3}}{2}$ because $\eta = 1, -\frac{1}{2} \pm i \frac{\sqrt{3}}{2}$.

When this is true, one has uniformly by (12) that $h(z) = az + b$ for $ac \neq 0$. As $\mathbf{p}(z)$ has two distinct zeros in \mathfrak{S} and thus in each associated lattice, we observe that all the zeros $\{z_j\}_{j=1}^{\infty}$ of $\mathbf{p}(z)$ are transferred to each other through (an integral multiple of) ac . We may for simplicity consider two cases where either $ac = \omega_1, \omega_2, \omega_1 + \omega_2$, or $ac \neq \omega_1, \omega_2, \omega_1 + \omega_2$ and $ac \in \mathfrak{S}$. The former cannot occur in view of (8) using the periodicity of \mathbf{p} and \mathbf{p}' , while the latter neither - noting $\mathbf{p}(z)$ has a unique double pole in each lattice, we substitute $z_{\infty} = -\frac{b}{a}$ into (8) to deduce

a contradiction

$$\infty = \frac{\eta \left\{ 1 - \frac{\sqrt{3}}{3} \mathbf{p}'(0) \right\}}{\mathbf{p}(0)} = \frac{\left\{ 1 + \frac{\sqrt{3}}{3} \mathbf{p}'(ac) \right\}}{\mathbf{p}(ac)} e^{\frac{\alpha c}{3}} < \infty.$$

Thus, $\mathbf{p}(h(a_{j,k} + c)) = 0$ may only occur for finitely many $a_{j,k}$'s. Without loss of generality, assume $\mathbf{p}(h(a_{j,k} + c)) \neq 0$ for each $k = 1, 2, \dots, \deg(h)$ and all $j > J$, with J being a sufficiently large positive integer. Since $\mathbf{p}(h(a_{j,k})) = 0$ and $(\mathbf{p}')^2(h(a_{j,k})) = -1$, one has $\mathbf{p}(h(a_{j,k} + c)) = \infty$ when $j > J$ by (8) again. As a consequence, noticing $O(\log r) = S(r, \mathbf{p}(h))$, we have

$$\begin{aligned} N\left(r, \frac{1}{\mathbf{p}(h(z))}\right) &\leq \bar{N}\left(r, \frac{1}{\mathbf{p}(h(z))}\right) + 2N\left(r, \frac{1}{h'(z)}\right) \\ &\leq \bar{N}(r, \mathbf{p}(h(z+c))) + 2T(r, h') + O(\log r) \leq \bar{N}(r, \mathbf{p}(h(z+c))) + S(r, \mathbf{p}(h)). \end{aligned} \quad (13)$$

Recall the first equality in (7) and estimate (10). One has

$$T(r, f) \leq T(r, \mathbf{p}(h)) + T(r, \mathbf{p}'(h)) + \frac{1}{3}T(r, e^{\alpha z}) + O(1) \leq O(T(r, \mathbf{p}(h))), \quad (14)$$

so that $\rho(f) = \rho(\mathbf{p}(h))$ and $S(r, f) = S(r, \mathbf{p}(h))$ from (11) and the side note after it.

Thus, $T(r, e^{\alpha z}) = S(r, f)$. As all the zeros of $f - e^{\frac{\alpha z + \beta}{3}}$, $f - \eta e^{\frac{\alpha z + \beta}{3}}$ and $f - \eta^2 e^{\frac{\alpha z + \beta}{3}}$ ($\eta \neq 1$) are of multiplicities at least 3 from (6), Yamanoi's second main theorem [18] yields

$$\begin{aligned} 2T(r, f) &\leq \sum_{m=1}^3 \bar{N}\left(r, \frac{1}{f - \eta^m e^{\frac{\alpha z + \beta}{3}}}\right) + \bar{N}(r, f) + S(r, f) \\ &\leq \frac{1}{3} \sum_{m=1}^3 N\left(r, \frac{1}{f - \eta^m e^{\frac{\alpha z + \beta}{3}}}\right) + N(r, f) + S(r, f) \leq 2T(r, f) + S(r, \mathbf{p}(h)). \end{aligned}$$

Therefore, one derives $T(r, f) = N(r, f) + S(r, \mathbf{p}(h))$ so that $m(r, f) = S(r, \mathbf{p}(h))$.

Finally, applying the lemma of logarithmic derivative, we have

$$m\left(r, \frac{1}{\mathbf{p}(h)}\right) \leq m(r, f) + m\left(r, \frac{\mathbf{p}'(h)h'}{\mathbf{p}(h)}\right) + S(r, \mathbf{p}(h)) = S(r, \mathbf{p}(h)) \quad (15)$$

again through the first equality in (7). Combining (13) and (15) leads to

$$\begin{aligned} T(r, \mathbf{p}(h)) + O(1) &= T\left(r, \frac{1}{\mathbf{p}(h)}\right) = m\left(r, \frac{1}{\mathbf{p}(h(z))}\right) + N\left(r, \frac{1}{\mathbf{p}(h(z))}\right) \\ &\leq \bar{N}(r, \mathbf{p}(h(z+c))) + S(r, \mathbf{p}(h)) \leq \frac{1}{2}N(r, \mathbf{p}(h(z+c))) + S(r, \mathbf{p}(h)) \\ &\leq \frac{1}{2}T(r, \mathbf{p}(h(z+c))) + S(r, \mathbf{p}(h)) \leq \frac{1}{2}T(r, \mathbf{p}(h)) + S(r, \mathbf{p}(h)), \end{aligned} \quad (16)$$

where theorem 2.1 of Chiang and Feng [5] was applied. This is a contradiction. \square

Example 4. Assume $f(z)$ is given by (7) through $h(z) = e^z$. Then, $\rho(f) = \infty$, and for $c = \pi i$ and each α with $e^{\alpha c} = 1$, $f^3(z) + f^3(z+c) = e^{\alpha z + \beta}$ for all $\beta \in \mathbf{C}$.

Example 5. Let $f_1(z) = e^{\frac{\alpha z + \beta}{2}} \sin(z)$ and $f_2(z) = e^{\frac{\alpha z + \beta}{2}} \sin(e^{4iz} + z)$. Then, $\rho(f_1) \leq 1$ and $\rho(f_2) = \infty$. For $c = \frac{\pi}{2}$ and each α with $e^{\alpha c} = 1$, $f_j^2(z) + f_j^2(z+c) = e^{\alpha z + \beta}$ for $j = 1, 2$ and all $\beta \in \mathbf{C}$.

Example 6. Define $f_1(z) = e^z + \frac{e^{\alpha z + \beta}}{2}$ and $f_2(z) = e^{e^{2z} + z} + \frac{e^{\alpha z + \beta}}{2}$. Then, $\rho(f_1) \leq 1$ and $\rho(f_2) = \infty$. For $c = i\pi$ and each α with $e^{\alpha c} = 1$, $f_j(z) + f_j(z+c) = e^{\alpha z + \beta}$ for $j = 1, 2$ and all $\beta \in \mathbf{C}$.

Opposite to theorem 2, even the existence of finite order or infinite order solutions to $f^2(z) + f^2(z+c) = e^{\alpha z + \beta}$ may be described for special α, c , I was not able to characterize systematically all possible solutions to this difference equation. The same concern occurs for the existence of infinite order solutions to $f^3(z) + f^3(z+c) = e^{\alpha z + \beta}$ in a systematic manner.

Finally, we briefly consider $f(z) + f(z+c) = e^{\alpha z + \beta}$. Recall (5) to choose $f(z) = de^{\alpha z + \beta}$ and $f(z+c) = e^{\alpha c}f(z)$. When $d(1+e^{\alpha c}) = 1$, we are done. I imagine the general solutions may be of the form $f(z) = \delta(z) + de^{\alpha z + \beta}$ for a meromorphic function $\delta(z)$ with $\delta(z+c) = -\delta(z)$ and $d(1+e^{\alpha c}) = 1$; in addition, the general solutions may be of the form $f(z) = \delta(z) - \frac{z}{c}e^{\alpha z + \beta}$ for $e^{\alpha c} = -1$. When $n \geq 2$, then equation (5) may have no solution if $e^{\alpha c} = -1$.

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